Exact Solutions of Gas-Particle Nozzle Flows

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An exact inverse method for the calculation of gas-particle flows, with arbitrary velocity and temperature lags, in rocket nozzles is presented In this method, the functional dependence of the particle velocity is assumed, and the corresponding nozzle shape and flow parameters are calculated It is shown that, for particle velocities of arbitrary complexities, the exact determination of the flow field and nozzle shape requires, at most, two quadratures The method is illustrated by an example

Introduction

THE development of high-energy propellants has increased the interest in estimating phase nonequilibrium effects on the performance of rocket engines In general, estimate of such an effect requires lengthy numerical calculations; this is because the location of the sonic point and the value of the critical mass flow rate are not known a priori [1] Thus, in order to study the influence of a given parameter on the performance, a great number of specific calculations are needed

Analytical solutions for the gas-particle nozzle-flow problem were given by Kliegel[2] and Rannie [3] Kliegel assumed that the particle kinetic and thermal lags are constant, whereas Rannie used a perturbation technique The object of this investigation is to add to these analytical solutions by outlining an inverse method that calculates for each particle velocity the flow parameters and a nozzle shape The various solutions resulting from the various particle velocities may then be used to estimate the effects of the various parameters on the performance of typical rocket nozzles

Analysis

Introducing the usual assumptions, the gas-particle nozzle-flow equations may be written as

$$\rho_g u_g A = \dot{w}_g \qquad \rho_p u_p A = \dot{w}_p \tag{1}$$

$$\dot{w}_g \frac{du_g}{dx} + \dot{w}_p \frac{du_p}{dx} + A \frac{dP_g}{dx} = 0 \tag{2}$$

$$\dot{w}_g[C_{pg}(T_g - T_{g0}) + \tfrac{1}{2}u_g^2] + \dot{w}_p[C_p(T_p - T_{p0}) + \tfrac{1}{2}u_p^2] = 0 \tag{3}$$

$$\frac{du_p}{dx} = B\left(\frac{u_g - u_p}{u_p}\right) \tag{4}$$

$$\frac{dT_p}{dx} = C\left(\frac{T_g - T_p}{u_p}\right) \tag{5}$$

$$P_g = R\rho_g T_g \tag{6}$$

Equations (1–3) are the equations of continuity, momentum, and energy; Eqs (4) and (5) are the particle momentum and energy equations; and Eq (6) is the equation of state In these equations the subscript g refers to the gas, and the subscript p refers to the particles; C_p indicates the specific heat of the particles; and B and C are constants The rest of the symbols have their usual meaning Since thermal equilibrium may be assumed in the chamber, T_{g0} and T_{p0} may be taken as equal

Letting

$$u_p = BK(x) \qquad T_p - T_{p0} = L(x) \tag{7}$$

Eqs (4) and (5) give

$$u_g = BK(1 + K') \qquad T_g - T_{g0} = (B/C)KL' + L \tag{8}$$

K and L are not independent, and a relation between them may be obtained by substituting Eqs (7) and (8) into Eq (3) Carrying out the indicated operation, one obtains

$$L' + (\lambda/K)L = -\beta K[\alpha + (1 + K')^2] \tag{9}$$

where

$$\lambda = \frac{C}{B}\left(1 + \alpha\frac{C_{ps}}{C_{pg}}\right) \qquad \beta = \frac{1}{2}\frac{BC}{C_{pg}} \qquad \alpha = \frac{\dot{w}_p}{\dot{w}_g} \tag{10}$$

For a given K, Eq (9) represents a first-order linear equation in L Thus L is given by

$$L = -\beta \exp[-\textstyle\int(\lambda/K)dx]\{\int \exp\int(\lambda/K)dx K[\alpha + (1 + K')^2]dx + \text{const}\} \tag{11}$$

and, therefore, the velocities and temperatures may be expressed in terms of K

An expression for the area is needed before one can proceed with the calculation of the other flow parameters Such an expression can be obtained from Eq (2) Using Eqs (1) and (6), one finds

$$A\frac{dP_g}{dx} = \frac{R\dot{w}_g T_g}{u_g}\frac{d}{dx}\left[\ln\left(\frac{T_g}{Au_g}\right)\right] \tag{12}$$

Hence, substituting Eq (12) into Eq (2), one obtains

$$\ln\frac{T_g}{Au_g} = -\frac{1}{R}\int\frac{u_g}{T_g}\left[\frac{du_g}{dx} + \alpha\frac{du_p}{dx}\right]dx + \text{const} \tag{13}$$

which gives an expression for A in terms of K This expression for A is used then in Eqs (1) to give ρ_g and ρ_p With ρ_g and T_g known, Eq (6) gives an expression for P_g

It is evident from Eqs (11) and (13) that, if $K(x)$ is a function of arbitrary complexity, the integration in Eqs (11) and (12) will have to be carried out numerically Such an integration should be straightforward

Illustrative Example

As an illustration of the foregoing procedure, a simple example is presented It can be shown that

$$K = K_0 x \tag{14}$$

results in the special case discussed by Kliegel, and, therefore, an alternative example will be selected The special case

$$K = K_0 x^{1/2} \tag{15}$$

will be discussed here Introducing the substitution

$$\xi = x^{1/2} \tag{16}$$

Eq (9) can be integrated to give

$$T_p - T_{p0} = L = -\beta(K_0^2/\lambda)\{(1 + \alpha)\xi^2 + K_0(1/\lambda)(\lambda - 1 - \alpha)\xi + (K_0^2/4\lambda^2)[\lambda^2 - 2\lambda + 2\alpha]\} \tag{17}$$

where, for the sake of simplicity, the value of $L(0)$ was assumed to result in a constant of integration of zero Using Eqs (15–17), Eqs (8) give

$$u_g = BK_0[\xi + (K_0/2)] \tag{18}$$

$$T_g = T_{g0} - (\beta K_0^2/\lambda)\{(1 + \alpha)\xi^2 + (K_0/\lambda)[\lambda - 1 - \alpha + (1 + \alpha)(BK_0\lambda/C)]\xi + (K_0^2/4\lambda^2)[\lambda^2 - 2\lambda + 2 + 2\alpha + 2(B/C)\lambda(\lambda - 1 - \alpha)]\} \equiv \lambda_1 + \lambda_2\xi + \lambda_3\xi^2 \tag{19}$$

Substitution of Eqs (15, 16, 18, and 19) into Eq (13) gives

$$\frac{A}{A_0} = \frac{T_g^{\mu}}{u_g}\exp\left[-\eta\tan^{-1}\left(\frac{2\lambda_3\xi + \lambda_2}{\zeta}\right)\right] \tag{20}$$

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where

$$\mu = 1 - \frac{\lambda B^2}{2\beta R}$$

$$\eta = \frac{2B^2K_0^2}{\zeta R}(1 + \alpha)\left(\frac{K_0}{2} - \frac{\lambda_2}{2\lambda_3}\right)$$

$$\zeta = (4\lambda_1\lambda_3 - \lambda_2^2)^{1/2} \qquad (21)$$

and A_0 is a constant of integration Equations (1) and (6) give ρ_p, ρ_g, and P_g

It may be concluded that, although this method does not give the exact solution to a specific problem directly, it is very useful in discussing the influence of the various parameters on performance

References

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Area Change with a Free-Piston Shock Tube

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THE free piston shock tube has been shown[1 2] to be a viable alternative to the arc-driven shock tube as a means of achieving shock Mach numbers substantially in excess of 20 in air It differs from the conventional shock tube in that the driver gas is initially contained in a large "compression tube" and, immediately prior to rupture of the shock-tube diaphragm, is heated and raised to high pressure through compression into a driver section of much smaller volume by motion of the free piston along this tube

To date, this modification has been considered only in conjunction with a constant-area shock tube The length of the shock-tube driver section should be at least a few times greater than the diameter to insure that the flow is not grossly distorted by the diaphragm opening process, and, therefore, when high volumetric compression ratios are desired in the driver gas, the diameter of the shock tube must be much smaller than the diameter of the compression tube Because reduction in diameter has severe adverse effects on shock-tube flow, this constitutes an important limitation on free-piston shock tubes In this note, it is shown that this limitation can, in principle, be removed by increasing the cross-sectional area in passing from the driver to the driven section, a technique initially employed by Lin and Fyfe[3] in a shock tube with combustion driver

The piston is sufficiently massive that it remains effectively stationary for the duration of the shock-tube flow, and so a wave diagram can be drawn, as in Fig 1, by following Lin and Fyfe In the usual notation, with $\gamma_4 = \gamma$, the driver-interface pressure ratio can, therefore, be written as

$$\frac{P_3}{P_4} = \left[f\left(1 - \frac{\gamma - 1}{2f}\frac{u_3}{a_4}\right)\right]^{2\gamma/(\gamma-1)} \qquad (1)$$

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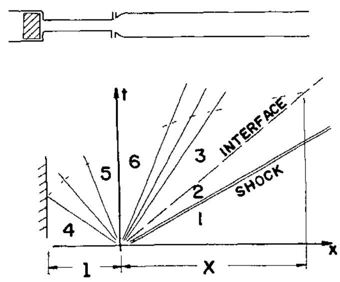

Fig 1 Wave diagram for shock-tube flow

where

$$f = \left(\frac{2}{\gamma + 1}\right)^{1/2}\left(1 + \frac{\gamma - 1}{2}M_6\right)\left(1 + \frac{\gamma - 1}{2}M_6^2\right)^{-1/2}$$

and

$$\frac{A_1}{A_4} = \frac{1}{M_6}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2}M_6^2\right)\right]^{(\gamma+1)/2(\gamma-1)}$$

M_6 is the flow Mach number at the downstream end of the area change From Eq (1), an expression may be obtained comparing driver-gas volumetric compression ratios in constant-area and change-of-area shock tubes, when both produce the same interface velocity (and hence the same shock Mach number) with the same pressure ratio P_4/P_3, i e ,

$$(R'/R)^{(\gamma-1)/2} = a_4'/a_4 = 1 - (1 - f)[1 - (P_3/P_4)^{(\gamma-1)/2\gamma}]^{-1}$$

where primed symbols indicate quantities associated with the constant-area shock tube, and R is the driver-gas volumetric compression ratio This is used to plot R/R' and l/l', where l is the length of the driver section, in Figs 2a and 2b, the curves being terminated when the available pressure ratio becomes insufficient to produce fully expanded flow through the area change

The limited volume allowed for the driver section implies that special attention must be paid to the distance X along the shock tube at which useful flow is terminated by the head of the expansion wave reflected from the upstream end of the driver overtaking the interface Calculation of the (x, t) equation for this "reflected head" is a straightforward procedure (e g , Ref 4), and, when it is combined with the (x, t) equation for the interface, it is found that

$$\frac{X}{l} = \frac{u_3}{a_4}(1 + M_6)\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2}M_6\right) \times \left(1 - \frac{\gamma - 1}{2f}\frac{u_3}{a_4}\right)\right]^{-(\gamma+1)/2(\gamma-1)}$$

or, comparing for identical performance as in the foregoing,

$$\frac{X}{X'} = \left(\frac{a_4'}{a_4}\right)^{(\gamma+1)/(\gamma-1)}\frac{1 + M_6}{2M_6} \times \left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2}M_6^2\right)\right]^{(\gamma+1)/4(\gamma-1)}$$

This ratio is plotted in Fig 2c

The curves show that, where a choice is possible, a constant-area shock tube is to be preferred to a change of area, since the latter requires a higher volumetric compression ratio to achieve the same performance and roughly the same duration of useful flow It is when constant area leads to a driver section that is too short that area change becomes important, enabling performances that could not otherwise be achieved For example, using a compression tube 10 ft long and 3 in in diameter with helium driver gas, Ref 2 indicates that a shock Mach number of 30 in air can be achieved, with $P_4/P_3 = 10^3$, in a 2-in -diam constant-area